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A stable approach to an unstable homotopy spectral sequence

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Abstract

Recently, Bendersky and Thompson introduced a spectral sequence which, for many spaces X, converges to the v_1-periodic homotopy groups of X. It is proved that the E_2-term of this spectral sequence is often given by Ext in the category of stable p-adic Adams modules of $QK^1(X;\mathbb{Z}_p^\wedge)/\mathrm{im}(\psi^p)$. We compute this spectral sequence when $p=2$ and X is the exceptional Lie group F_4, yielding as a new result the 2-primary v_1-periodic homotopy groups of F_4. Some new general results about convergence of this spectral sequence are also proved.

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1. Statement of results

In [10], a spectral sequence, which we call the BTSS, was constructed for simply connected spaces X; it converges, on a class of spaces which includes finite H-spaces and K_*-strongly spherically resolved spaces, to the homotopy groups of the K-completion of X, denoted $\hat{X}$. This convergence, and other convergence issues, will be discussed in Section 5.

We will work with the v_1-periodic version of this spectral sequence, localized at any prime p, although the main thrust of this paper is the case $p=2$. Our main result, Theorem 1.1, shows that the E_2-term of the BTSS, denoted $E_2(X)$, can, for many spaces X, be computed directly from the indecomposables $QK^1(X;\mathbb{Z}_p^\wedge)$ and the Adams operations ψ^k. This should be contrasted with the method used in [10] to compute E_2, which involved delicate manipulations with the unstable cobar complex.

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Let $\mathscr{A}$ denote the abelian category of stable p-adic Adams modules [17, 2.6] An object in $\mathscr{A}$ is a p-profinite abelian group with Adams operations ψ^k for $k \in \mathbb{Z} - p\mathbb{Z}$, satisfying certain axioms. Our main theorem applies to simply connected spaces X for which there is a torsion-free K_*K-subcomodule $M \subset PK_{\mathrm{odd}}(X;\mathbb{Z})$ such that $K_*(X;\mathbb{Z}) \approx \Lambda(M)$ as ($\mathbb{Z}$-graded) $K_*(K)$-coalgebras, while the $\mathbb{Z}/2$-graded $K^*(X;\mathbb{Z}_p^\wedge)$ is isomorphic to $\hat{\Lambda}(M_1 \otimes \mathbb{Z}_{p^\infty})^\#$ with ψ^p monic on $QK^1(X;\mathbb{Z}_p^\wedge)$. Here $(-)^\#$ denotes Pontrjagin duality, $P(-)$ denotes the primitives in a coalgebra, and Λ an exterior algebra. We prove in Proposition 5.5 that simply connected mod p finite H-spaces whose rational homology is associative and K_*-strongly spherically resolved spaces (see Definition 5.3) satisfy these conditions.

Theorem 1.1. *If X is a space satisfying the above conditions, then the E_2-term of the* BTSS *satisfies*

$$E_2^{s,t}(X) \approx \begin{cases} \mathrm{Ext}^s_{\mathscr{A}}(QK^1(X;\mathbb{Z}_p^\wedge)/\mathrm{im}(\psi^p), QK^1(S^t;\mathbb{Z}_p^\wedge)) & \text{if } t \text{ is odd}, \\ 0 & \text{if } t \text{ is even}. \end{cases}$$

We prove Theorem 1.1 in Section 2. In Section 3, we develop a method of computing $\mathrm{Ext}_{\mathscr{A}}(-,-)$, especially when $p=2$, which case is much more delicate than the odd-primary. These results build upon earlier work of Bousfield [17,13,14] and Bendersky and Davis [7].

In Section 4, we apply these results to compute the BTSS and v_1-periodic homotopy groups of the exceptional Lie group F_4. Determination of d_3-differentials requires comparison with d_3 in other spaces related to F_4 by fibrations. We obtain

Theorem 1.2. *Let $e = \min(12, 2v(i-3)+8)$. Then the 2-primary v_1-periodic homotopy groups of F_4 are given by*

$$v_1^{-1}\pi_{8i+d}(F_4;2) \approx \begin{cases} \mathbb{Z}/2^e, & d=-3, \\ \mathbb{Z}/2^e \oplus \mathbb{Z}/2, & d=-2, \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2, & d=-1,0, \\ \mathbb{Z}/2^6 \oplus \mathbb{Z}/2, & d=1, \\ \mathbb{Z}/2^6, & d=2, \\ 0, & d=3,4. \end{cases}$$

Here, and throughout, $v(-)$ denotes the exponent of 2 in an integer. The $2v(-)$ occurring in the answer is a surprise, compared to previous computations for other Lie groups.

Theorem 1.2 left $SO(n)$, E_6, E_7, and E_8 as the only compact simple Lie groups whose 2-primary v_1-periodic homotopy groups have not been computed. Davis in [19] completed the computation of all odd-primary v_1-periodic homotopy groups of all compact simple Lie groups. While this paper was being considered, Bendersky and Davis used Theorem 1.1 to compute $v_1^{-1}\pi_*(SO(n);2)$ in [9], and Davis used it to compute $v_1^{-1}\pi_*(E_6;2)$ in [21]. Thus only E_7 and E_8 remain, and the E_2-term of their BTSS has been computed using Theorem 1.1.

The distinction between $v_1^{-1}\pi_*X$, which we want, and $v_1^{-1}\pi_*\hat{X}$, which the BTSS often computes, is presented in the following definition.

Definition 1.3. A space X satisfies the Completion Telescope Property (CTP) if the natural map $X \to \hat{X}$ induces an isomorphism in $v_1^{-1}\pi_*(-)$.

It follows easily from [10, 4.12]; [11, 1.4] that S^{2n+1} and ΩS^{2n+1} satisfy the CTP. In Section 5, we prove

Theorem 1.4. *The spaces S^{2n}, ΩS^{2n}, G_2, and F_4 satisfy the* CTP.

2. Proof of Theorem 1.1

We shall adopt the following notation and conventions. $K^*(-)$ will denote $\mathbb{Z}/2$-graded K-cohomology with $\mathbb{Z}_p^\wedge$ coefficients, while $K_*(-)$ denotes $\mathbb{Z}$-graded K-homology with $\mathbb{Z}$ coefficients. This difference in gradings is primarily for convenience of exposition: K-homology needs $\mathbb{Z}$-gradings for its unstable condition, while our use of K-cohomology is primarily in $K^1(-)$. The Bott element $v_1 \in K_2 \approx K^{-2}$ gives isomorphisms

$$K_i(X) \xrightarrow{v_1} K_{i+2}(X), \quad K^i(X) \xrightarrow{v_1} K^{i-2}(X)$$

for all integers i, allowing passage between $\mathbb{Z}/2$-graded and $\mathbb{Z}$-graded theories. Coactions $K_*X \xrightarrow{\psi} K_*K \otimes K_*X$ and Adams operations ψ^k in $K^*(X)$ are passed along by $\psi(v_1x) = v_1\psi(x)$ and $\psi^k(v_1x) = kv_1\psi^k(x)$.

Note that if $K_i(X)$ is torsion-free, then $K^i(X)$ and $K_i(X) \otimes \mathbb{Z}_{p^\infty}$ are Pontrjagin dual to one another. We will denote $(M \otimes \mathbb{Z}_{p^\infty})^\#$ by $M^\#$ for notational simplicity.

Recall that a profinite abelian group or K^*-module is a *stable p-adic Adams* module if it admits operations ψ^k for $k \in \mathbb{Z} - p\mathbb{Z}$ satisfying the properties of [17, 2.6]. A *p-adic Adams* module admits operations ψ^k for all $k \in \mathbb{Z}$ as in [17, 2.8]. If M is a stable p-adic Adams module, then the free p-adic Adams module, $\tilde{F}(M)$, generated by M is defined as follows.

Definition 2.1 (Bousfield [17, 3.1]). As abelian groups or K^*-modules

$$\tilde{F}(M) = M \times M \times \cdots,$$

with Adams operations defined by

$$\psi^k(x_1, x_2, \ldots) = \begin{cases} (\psi^k x_1, \psi^k x_2, \ldots) & \text{if } k \not\equiv 0 \bmod p, \\ (0, x_1, x_2, \ldots) & \text{if } k = p. \end{cases}$$

The following result plays a key role in the proof of Theorem 1.1.

Proposition 2.2. *There is an isomorphism of $\mathbb{Z}/2$-graded p-adic Adams modules*

$$QK^*(SU) \approx \tilde{F}(\Gamma),$$

where Γ is a projective object of $\mathscr{A}$ on one generator of grading 1.

Proposition 2.2 is a special case of [17, 3.3]. To see this, we let $E=(\mathbf{K}\wedge S^1)\langle 2\rangle$, the 1-connected cover of $\mathbf{K}\wedge S^1$ localized at $\mathbb{Z}/p$. We have $\Omega^\infty E=SU=U\langle 2\rangle$. Theorem 3.3 of [17] asserts that

$$K^*(\Omega^\infty E)\approx \hat{\Lambda}\tilde{F}K^1(E). \tag{2.3}$$

Since $K^1(E)\approx\Gamma$ by Bousfield [16, p. 24], Proposition 2.2 follows from (2.3).

We present the following alternative proof.

Alternate proof of Proposition 2.2. We have $QK^*(SU)=\tilde{K}^*(\Sigma CP^\infty)=K^*\{\xi_1,\xi_2,\ldots\}$, the free K^*-module with basis $\xi_k=\xi^k-1$, with ξ the canonical line bundle over CP^∞. Note that ξ_k has grading 1 in $K^*(SU)$. The Adams operations act by $\psi^r\xi_k=\xi_{rk}$ for $r\geqslant 0$. For $a\geqslant 0$, define M_a to be the K^*-submodule of $K^*\{\xi_1,\xi_2,\ldots\}$ generated by $\{\xi_{p^ak}\,|\,(k,p)=1\}$. Denote M_0 by Γ.

We claim that Γ is a projective stable p-adic Adams module on one generator. First note that Γ admits Adams operations ψ^k for positive k prime to p. Since $\Gamma\approx K^*(CP^\infty)/\mathrm{im}(\psi^p)$, it also admits the operation ψ^{-1}, since $\psi^{-1}(\mathrm{im}(\psi^p))\subset\mathrm{im}(\psi^p)$. Next observe that Γ has one generator as stable Adams module since $\xi_k=\psi^k(\xi_1)$ for positive k prime to p. Finally, to show that Γ is projective, let $B\xrightarrow{\phi}C$ be a surjection of stable Adams modules and $\Gamma\xrightarrow{g}C$ a morphism of stable Adams modules. Define $\Gamma\xrightarrow{\bar{g}}B$ by $\bar{g}(\xi_1)=b_0$ for some b_0 satisfying $\phi(b_0)=g(\xi_1)$ and $\bar{g}(\xi_k)=\psi^kb_0$ for positive k prime to p. We must show that $\bar{g}$ also respects the action of ψ^{-1}.

For this, note that the p-adic stable Adams modules Γ and B are inverse limits of finite Adams modules. By a property of stable p-adic Adams modules [17, 2.6], for each n, there exists m such that $\psi^k\equiv\psi^{k+m}\bmod p^n$ in Γ and B for all integers k. Thus $\bar{g}$ commutes with $\psi^{-1}\bmod p^n$ for all positive integers n. Passing to the inverse limit shows that $\bar{g}$ respects the action of ψ^{-1}.

The map $\Gamma\to M_a$ defined on generators by $\xi_k\mapsto\xi_{p^ak}$ is an isomorphism of stable p-adic Adams modules. Thus

$$QK^*(SU)=M_0\times M_1\times\cdots\approx\Gamma\times\Gamma\times\cdots.$$

The action of ψ^p on $\Gamma\times\Gamma\times\Gamma\times\cdots$ is given on generators by

$$\psi^p(\xi_{k_1},\xi_{k_2},\ldots)=(0,\xi_{k_1},\xi_{k_2},\ldots).$$

Here, we have used that $\psi^p(\xi_k)$ in the ith factor corresponds to $\psi^p(\xi_{kp^i})=\xi_{kp^{i+1}}$, which is ξ_k in the $(i+1)$st factor. Hence $QK^*(SU)\approx\tilde{F}(\Gamma)$, yielding the result. $\square$

We denote by $\mathscr{M}$ the category of free ($\mathbb{Z}$-graded) K_*-modules, and by $\mathscr{S}$ the homotopy category of topological spaces. We recall the definition [10] of the functor V from $\mathscr{M}$ to itself, and the $\mathscr{V}$-resolution of certain $M\in\mathscr{M}$.

$$M\to V(M)\to V(VM)\to V(V^2M)\to\cdots. \tag{2.4}$$

To define $V(M)$, we first let $\mathbf{K}M$ be the spectrum realizing the homology theory $K_*(-;M)$. We then define KM to be $\Omega^\infty\mathbf{K}M$. Note that $\pi_*(KM)\approx M$, $*\geqslant 0$. For a space X with free K_*-homology, KX is defined to be $KK_*(X)$ (equivalently $KX=\Omega^\infty(\mathbf{K}\wedge\Sigma^\infty X)$). We use K to denote both the functor $\mathscr{M}\to\mathscr{S}$ and the functor $\mathscr{S}\to\mathscr{S}$. Then $V(M)$ is defined to be the indecomposable quotient $Q(K_*KM)$. This V is the functor of a cotriple on $\mathscr{M}$, which means that there are natural transformations $\delta:V\to V^2$ and $\varepsilon:V\to I$ satisfying certain identities. (See [4, 5.2].)

Note that if all basis elements of M have odd dimension, then the same is true of $V(M)$. This follows since $K_r = U$ if r is odd, and $K_*(U)$ is generated by odd-dimensional elements. The 0-part of Theorem 1.1 now follows from (2.7), (2.10), and (2.11).

The category $\mathscr{V}$ of unstable K_*K-comodules consists of objects $M \in \mathscr{M}$ equipped with a K_*-homomorphism $\eta_M : M \to V(M)$ with the usual commutative diagrams [2, 2.15]. If X is as in Theorem 1.1, $K_*(X)$ is a Hopf algebra with $M = PK_*(X) = QK_*(X) \in \mathscr{M}$. The unit map $h : X \to KX = KK_*(X)$ induces the unstable coaction, $K_*(h) : K_*(X) \to K_*(KX)$, which in turn induces a morphism

$$\eta_M : M = PK_*(X) \to PK_*(KX) \to PK_*(KQK_*X) = PK_*(KM) = V(M), \tag{2.5}$$

which gives M the structure of an unstable K_*K-comodule. The map $KX \to KQK_*X$ which induces the second homomorphism comes from $KX = KK_*X$ and the natural morphism $K_*X \to QK_*X$. The last equality follows because $PK_*(KM) = QK_*(KM)$ if M is generated by odd-dimensional classes.

There are two maps in $\mathscr{V}$ from $V(M) \to V(V(M))$, namely $V(\eta_M)$ and $\eta_{V(M)}$. In general, there are $n+1$ coface maps $V(V^{n-1}(M)) \to V(V^n(M))$. There is also the map $V(V(M)) \to V(M)$, which is not in $\mathscr{V}$. In general, there are n codegeneracy maps $V(V^n(M)) \to V(V^{n-1}(M))$. These maps fit together to generate the $\mathscr{V}$ cosimplicial *resolution* $\mathscr{C}$ (we omit the codegeneracies):

$$M \to V(M) \rightrightarrows V^2(M) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots. \tag{2.6}$$

The coboundary maps in resolution (2.4) are the alternating sums of the coface maps in (2.6). Note that, whereas each $V^s(M)$ satisfies $V^s(M)_t \approx V^s(M)_{t+2}$ for all t, the coboundary maps in (2.4) do not share this period-2 behavior, since they do not commute with the periodicity operator. That is, the groups can be considered as being $\mathbb{Z}/2$-graded, but the morphisms cannot.

As usual, $\mathrm{Ext}_{\mathscr{V}}$ is defined as the derived functors of $\mathrm{Hom}_{\mathscr{V}}$:

$$\begin{aligned}\mathrm{Ext}_{\mathscr{V}}^{s,t}(K_*, M) &= H_s(\mathrm{Hom}_{\mathscr{V}}(K_*(S^t), \mathscr{C}))\\ &= H_*(M_t \to V(M)_t \to V^2(M)_t \to \cdots),\end{aligned} \tag{2.7}$$

where the coboundary maps in (2.7) are the alternating sums of the coface maps in

$$M \rightrightarrows V(M) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots, \tag{2.8}$$

which is obtained by applying the adjointness isomorphism

$$\mathrm{Hom}_{\mathscr{V}}(K_*(S^t), V(N)) = N_t \tag{2.9}$$

to (2.6).

The connection with $E_2(X)$ is given by applying the free commutative algebra functor F to resolution (2.6) to obtain a resolution $F(\mathscr{C})$ of $K_*(X)$ by injectives in the non-abelian category $\mathscr{G}$ introduced in [4, Section 6]. We are using the fact that $FV(M) = K_*KM$, which are the injectives in the category $\mathscr{G}$. Applying $\mathrm{Hom}_{\mathscr{G}}(K_*(S^t), -)$ to $F(\mathscr{C})$ also gives (2.8). Thus

$$\mathrm{Ext}_{\mathscr{G}}^{s}(K_*S^t, K_*X) \approx \mathrm{Ext}_{\mathscr{V}}^{s}(K_*S^t, PK_*X) \tag{2.10}$$

if X is as in Theorem 1.1. From [10, 4.3], we have

$$E_2^{s,t}(X) = \mathrm{Ext}_{\mathscr{G}}^{s}(K_*(S^t), K_*X) \quad \text{for } t-s>0. \tag{2.11}$$

If N is a free K_*-module with basis B and $N_{\mathrm{ev}} = 0$, then $V(N) = \oplus_{b \in B} PK_*K_b$ (recall $PK_*K_i = QK_*K_i$ if i is odd). Each K_b is a copy of $U = KS^{|b|}$, the 0-space in the Ω-spectrum of $\mathbf{K} \wedge \Sigma^\infty S^{|b|}$. The

Pontryagin dual $V(N)^{\#}$ is isomorphic to $QK^*(KN)$, which gives it the structure of p-adic Adams module. We restrict attention to the $\mathbb{Z}/2$-graded module, and note that it is 0 in grading 0. Using Proposition 2.2, we obtain

$$\begin{aligned} V(N)_1^{\#} &\approx \bigoplus_{b\in B} QK^1(S^1 \times SU_{(b)}) \\ &\approx \bigoplus_b (K^1(S^1)_b \times \tilde{F}(\Gamma)_b) \\ &\approx \bigoplus_b K^1(S^1)_b \times \tilde{F}(\Gamma \otimes N_1^{\#}), \end{aligned} \tag{2.12}$$

where $\Gamma \otimes N_1^{\#}$ has the extended stable Adams module structure, i.e. for k prime to p, $\psi^k(\gamma \otimes n) = \psi^k(\gamma) \otimes n$. Here Γ has grading 0, since it is tensored with classes of grading 1. We define $\tilde{V}(N)_1^{\#}$ to be $\tilde{F}(\Gamma \otimes N_1^{\#})$. We remind the reader that in this paragraph and throughout the remainder of this section $L^{\#}$ means $(L \otimes \mathbb{Z}_{p^\infty})^{\#}$ for any abelian group L.

Now we deduce our main theorem.

Proof of Theorem 1.1. Let X and M be as in Theorem 1.1 and its preamble. The Pontrjagin dual of the complex obtained by tensoring (2.4) with $\mathbb{Z}_{p^\infty}$,

$$0 \leftarrow M^{\#} \leftarrow V(M)^{\#} \leftarrow V(VM)^{\#} \leftarrow \cdots \tag{2.13}$$

is acyclic. The maps in complex (2.13) are in the category of p-adic Adams modules. This follows from dualizing (2.5). In particular, the following diagram of exact sequences commutes.

$$\begin{array}{ccccccc}
0 \leftarrow & M^{\#} & \leftarrow & (V(M))^{\#} & \leftarrow & (V(VM))^{\#} & \leftarrow \cdots \\
& \downarrow{\scriptstyle \psi^p} & & \downarrow{\scriptstyle \psi^p} & & \downarrow{\scriptstyle \psi^p} & \\
0 \leftarrow & M^{\#} & \leftarrow & (V(M))^{\#} & \leftarrow & (V(VM))^{\#} & \leftarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \leftarrow & M^{\#}/\operatorname{im}(\psi^p) & \leftarrow & (V(M))^{\#}/\operatorname{im}(\psi^p) & \leftarrow & (V(VM))^{\#}/\operatorname{im}(\psi^p) & \leftarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
& 0 & & 0 & & 0 &
\end{array}$$

Since ψ^p is injective, the vertical sequences of the above diagram are short exact. The induced long exact sequence in homology implies the bottom row is a resolution of $M^{\#}/\operatorname{im}(\psi^p)$. Now ψ^p is an isomorphism on the factors of $K^*(S^1)$, and by (2.12) and Definition 2.1 there is an isomorphism of stable p-adic Adams modules $(\tilde{V}(V^sM))^{\#}/\operatorname{im}(\psi^p) \approx \Gamma \otimes (V^sM)^{\#}$. So the bottom row is a resolution of $M^{\#}/\operatorname{im}(\psi^p)$ by $\mathscr{A}$-projectives.

The boundary $d_s : V^s(M) \to V^{s+1}(M)$ in resolution (2.4) satisfies $d_s = V(d_{s-1}) - \eta_{V^s(M)}$. We wish to show that the following diagram commutes, where t is a positive odd integer.

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{V}}(K_*S^t, V^s(M)) & \xrightarrow{(d_s)_*} & \mathrm{Hom}_{\mathcal{V}}(K_*S^t, V^{s+1}(M)) \\
\downarrow \approx & & \downarrow \approx \\
\mathrm{Hom}_{K_*}(K_*S^t, V^{s-1}(M)) & & \mathrm{Hom}_{K_*}(K_*S^t, V^s(M)) \\
\downarrow \approx & & \downarrow \approx \\
\mathrm{Hom}_{AbGp}(V^{s-1}(M)_1^\#, K^1S^t) & & \mathrm{Hom}_{AbGp}(V^s(M)_1^\#, K^1S^t) \\
\downarrow \approx & & \downarrow \approx \\
\mathrm{Hom}_{\mathcal{A}}(V^s(M)_1^\#/\,\mathrm{im}(\psi^p), K^1S^t) & \xrightarrow{(d_s^\#)^*} & \mathrm{Hom}_{\mathcal{A}}(V^{s+1}(M)_1^\#/\,\mathrm{im}(\psi^p), K^1S^t)
\end{array} \tag{2.14}$$

The first of the vertical isomorphisms is due to (2.9). The second of the vertical isomorphisms is Pontrjagin duality. The third of the vertical isomorphisms is a consequence of

$$\begin{aligned}
V^s(M)^\#/\mathrm{im}(\psi^p) &\approx \tilde{V}(V^{s-1}M)^\#/\mathrm{im}(\psi^p) \approx \tilde{F}(\Gamma \otimes (V^{s-1}M)^\#)/\mathrm{im}(\psi^p) \\
&\approx \Gamma \otimes (V^{s-1}M)^\#
\end{aligned}$$

with Γ projective in $\mathscr{A}$ on one generator.

For the $V(d_{s-1})$ portion of d_s, commutativity of (2.14) is true because $(d_{s-1})_*$ and $(d_{s-1}^\#)^*$ can be placed as intermediate horizontal arrows, yielding three commutative squares. Commutativity of the $\eta_{V^s(M)}$ portion of (2.14) is proved using consideration of the unstable cobar complex, which we now describe.

The way in which $\mathrm{Ext}_{\mathscr{V}}(-)$ has been computed in papers such as [10,4,2] is by viewing V^sM as the subset of

$$\overbrace{E_*E \otimes_{E_*} \cdots \otimes_{E_*} E_*E}^{s} \otimes_{E_*} E_*M$$

satisfying an "unstable condition." Here E is a spectrum such as K or BP, and M is an unstable E_*E-comodule which is free as an E_*-module. Under this identification, $\eta_{VN} : VN \to V^2N$ sends $\gamma \otimes n$ to $\psi(\gamma) \otimes n$, and in the commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{V}}(K_*S^t, VN) & \xrightarrow{\eta_{VN*}} & \mathrm{Hom}_{\mathcal{V}}(K_*S^t, V^2N) \\
\downarrow \approx & & \downarrow \approx \\
\mathrm{Hom}_{K_*}(K_*S^t, N) & \longrightarrow & \mathrm{Hom}_{K_*}(K_*S^t, VN) \\
\downarrow \approx & & \downarrow \approx \\
N_t & \xrightarrow{\phi} & (VN)_t
\end{array}$$

the corresponding morphism ϕ sends n to $1 \otimes n$. Here we are thinking of N as $V^{s-1}M$. Similarly, with ϕ as above, there is a commutative diagram

$$\begin{tikzcd}
\Gamma \otimes N_1^{\#} & \Gamma \otimes (VN)_1^{\#} \arrow[l, "\Gamma\otimes\phi^{\#}"'] \\
\widetilde{F}(\Gamma \otimes N_1^{\#})/\operatorname{im}(\psi^p) \arrow[u, "\approx"'] & \widetilde{F}(\Gamma \otimes (VN)_1^{\#})/\operatorname{im}(\psi^p) \arrow[l] \arrow[u, "\approx"'] \\
(VN)_1^{\#}/\operatorname{im}(\psi^p) \arrow[u, "\approx"'] & (V^2N)_1^{\#}/\operatorname{im}(\psi^p) \arrow[l, "\eta_{VN}^{\#}"'] \arrow[u, "\approx"']
\end{tikzcd}$$

With $\operatorname{Hom}_{\mathscr{A}}(-, K^1S^t)$ applied to the second, these diagrams imply commutativity of the η_{V^sM} portion of (2.14) and hence of the diagram itself.

Thus the homology of the $(\operatorname{Hom}_{\mathscr{V}}(K_*S^t, V^sM), (d_s)_*)$-sequence is isomorphic to the homology of the $(\operatorname{Hom}_{\mathscr{A}}(V^sM_1^{\#}/\operatorname{im}(\psi^p), K^1S^t), (d_s^{\#})^*)$-sequence. These are the two groups which the theorem asserts to be isomorphic. $\square$

The following example might be instructive. Note that $X = S^{2n+1}$ satisfies the conditions of Theorem 1.1. In this case, $M^{\#} = M_n$, a free K^*-module on a single generator with $\psi^k = k^n$. The short exact sequence

$$0 \to M_n \xrightarrow{\psi^p} M_n \to M_n/\operatorname{im}(\psi^p) \to 0$$

induces an exact (Bockstein) sequence in $\operatorname{Ext}_{\mathscr{A}}$, which relates the unstable E_2 for S^{2n+1} with the stable E_2 for the sphere spectrum. Here $\operatorname{Ext}_{\mathscr{A}}(M_n)$ is the E_2-term of a K-based spectral sequence, indexed so as to converge to the *stable* v_1-periodic homotopy groups of S^{2n+1}.

3. Computing $\operatorname{Ext}_{\mathscr{A}}(-,-)$

In this section, we develop a method of computing $\operatorname{Ext}_{\mathscr{A}}^{s,t}(M)$ for a stable p-adic Adams module M. For simplicity of exposition, we focus mostly on modules in which $\psi^{-1} = -1$, which is all we need in this paper. The general case, described in Theorem 3.10, requires only minor modifications.

If $t = 2n+1$, we let $\operatorname{Ext}_{\mathscr{A}}^{s,t}(M) = \operatorname{Ext}_{\mathscr{A}}^{s}(M, S_t)$, where $S_t = QK^1(S^t; \mathbb{Z}_p^{\wedge})$ is $\mathbb{Z}_p^{\wedge}$ with $\psi^k = \cdot k^n$. In this section, $(-)^{\#}$ denotes ordinary Pontrjagin duality.

Theorem 3.1. *Let M be a finite stable p-adic Adams module with $\psi^{-1} = -1$.*

(a) *If p is odd and r denotes a generator of $(\mathbb{Z}/p^2)^{\times}$, then*

$$\operatorname{Ext}_{\mathscr{A}}^{s,2n+1}(M)^{\#} \approx \begin{cases} M/\operatorname{im}(\psi^r - r^n), & s = 1, \\ \ker((\psi^r - r^n)|M), & s = 2, \\ 0, & \textit{otherwise}. \end{cases}$$

(b) *Let $p = 2$, $M_2 = \ker(2|M)$, and $\theta = \psi^3 - 1$. If n is odd, there is an isomorphism*

$$\operatorname{Ext}_{\mathscr{A}}^{1,2n+1}(M)^{\#} \approx \operatorname{coker}((\psi^3 - 3^n)|M)$$

and a split short exact sequence

$$0 \to \operatorname{coker}(\theta|M/2) \to \operatorname{Ext}_{\mathscr{A}}^{2,2n+1}(M)^{\#} \to \ker((\psi^3 - 3^n)|M) \to 0.$$

If n is even, then

$$\operatorname{Ext}_{\mathscr{A}}^{1,2n+1}(M)^{\#} \approx \operatorname{coker}(\theta|M/2).$$

If $s+n$ is odd and $s>2$, there is a split short exact sequence

$$0 \to \operatorname{coker}(\theta|M/2) \to \operatorname{Ext}_{\mathscr{A}}^{s,2n+1}(M)^{\#} \to \ker(\theta|M_2) \to 0.$$

If $s+n$ is even and $s>1$, there is a split short exact sequence

$$0 \to \operatorname{coker}(\theta|M_2) \to \operatorname{Ext}_{\mathscr{A}}^{s,2n+1}(M)^{\#} \to \ker(\theta|M/2) \to 0.$$

The case $s=1$ was proved in [7]. The odd-primary case is proved in [17, Section 8].

The proof of Theorem 3.1 when $p=2$ will utilize the following elementary result. As in [14, Section 3], let *Inv* denote the category of abelian groups with involution ψ^{-1}. Let $M^{(\varepsilon)}$ denote a 2-local abelian group M with $\psi^{-1}=\varepsilon$.

Proposition 3.2. *If M is a 2-local abelian group with $\psi^{-1}=-1$, then*

$$\operatorname{Ext}_{Inv}^{s}(\mathbb{Z}_{(2)}^{((-1)^n)}, M) \approx \begin{cases} M_2 & \text{if } s+n \text{ even and } s \geqslant 0, \\ M/2 & \text{if } s+n \text{ odd and } s>0, \\ M & \text{if } s=0 \text{ and } n \text{ odd}. \end{cases}$$

Proof. Let P denote the object of *Inv* which is $\mathbb{Z}_{(2)} \oplus \mathbb{Z}_{(2)}$ with ψ^{-1} switching the summands. Note that P is projective. Let $\varepsilon = (-1)^n$. A projective resolution of $\mathbb{Z}_{(2)}^{(\varepsilon)}$ is given by

$$\cdots \xrightarrow{d_2} C_2 \xrightarrow{d_1} C_1 \xrightarrow{d_0} C_0 \to \mathbb{Z}_{(2)}^{(\varepsilon)} \to 0$$

with each $C_i = P$ and $d_i = 1 + (-1)^{i+1}\varepsilon\psi^{-1}$. The complex

$$\operatorname{Hom}_{Inv}(C_0, M) \xrightarrow{d_0^*} \operatorname{Hom}_{Inv}(C_1, M) \xrightarrow{d_1^*} \operatorname{Hom}_{Inv}(C_2, M) \xrightarrow{d_2^*} \cdots$$

is isomorphic to

$$M \xrightarrow{1+\varepsilon} M \xrightarrow{1-\varepsilon} M \xrightarrow{1+\varepsilon} \cdots$$

and the homology of this is as claimed in the proposition. □

Proof of Theorem 3.1. Let $\mathscr{G}\,Inv$ denote the category of 2-profinite abelian groups with involution. Similarly to [17, 8.3], we have

Proposition 3.3. *If M and N are stable 2-adic Adams modules, there is a natural exact sequence*

$$0 \to \operatorname{Hom}_{\mathscr{A}}(M,N) \to \operatorname{Hom}_{\mathscr{G}\,Inv}(M,N) \xrightarrow{\psi_M^3 - \psi_N^3} \operatorname{Hom}_{\mathscr{G}\,Inv}(M,N)$$

$$\to \operatorname{Ext}_{\mathscr{A}}^{1}(M,N) \to \operatorname{Ext}_{\mathscr{G}\,Inv}^{1}(M,N) \xrightarrow{\psi_M^3 - \psi_N^3} \operatorname{Ext}_{\mathscr{G}\,Inv}^{1}(M,N)$$

$$\to \mathrm{Ext}^2_{\mathscr{A}}(M,N) \to \mathrm{Ext}^2_{\mathscr{G}\,Inv}(M,N) \overset{\psi^3_M-\psi^3_N}{\to} \mathrm{Ext}^2_{\mathscr{G}\,Inv}(M,N)$$
$$\to \mathrm{Ext}^3_{\mathscr{A}}(M,N) \to \cdots .$$

The exact sequence of Proposition 3.3 is obtained from a short exact sequence

$$0 \to U(M) \overset{U\psi^3-\psi^3}{\to} U(M) \to M \to 0,$$

where $U : \mathscr{G}\,Inv \to \mathscr{A}$ is left adjoint to the forgetful functor. This U is a profinite version of the functor of [14, 6.6], and satisfies $\mathrm{Ext}^s_{\mathscr{A}}(U(M),N) \approx \mathrm{Ext}^s_{\mathscr{G}\,Inv}(M,N)$. Its existence, adjointness, exactness, and the fundamental short exact sequence above all follow by Pontrjagin duality from the analogous results for the functor $\bar{U}$ in [14, pp. 145–146].

Since $\mathscr{G}\,Inv$ is dual to the torsion subcategory of Inv, we have

$$\mathrm{Ext}^s_{\mathscr{G}\,Inv}(M,N) \approx \mathrm{Ext}^s_{Inv}(N^\#,M^\#). \tag{3.4}$$

If $N = QK^1(S^{2n+1})^\wedge$, then, since the Pontrjagin dual of $\mathbb{Z}_2^\wedge$ is $(\mathbb{Q}/\mathbb{Z})_{(2)}$, we obtain

$$\mathrm{Ext}^{s,2n+1}_{\mathscr{G}\,Inv}(M) \approx \mathrm{Ext}^{s-1}_{Inv}(\mathbb{Z}_{(2)}^{((-1)^n)}, M^\#)$$

by Proposition 3.9 and the Ext-sequence induced from

$$0 \to \mathbb{Z}_{(2)} \to \mathbb{Q}_{(2)} \to \mathbb{Q}/\mathbb{Z}_{(2)} \to 0. \tag{3.5}$$

If n is odd, the exact sequence of Proposition 3.3 becomes, using Proposition 3.2,

$$0 \to \mathrm{Ext}^{0,2n+1}_{\mathscr{A}}(M) \to 0 \to 0 \to \mathrm{Ext}^{1,2n+1}_{\mathscr{A}}(M) \to M^\# \overset{\psi^3-3^n}{\to} M^\#$$
$$\to \mathrm{Ext}^{2,2n+1}_{\mathscr{A}}(M) \to M_2^\# \overset{\psi^3-1}{\to} M_2^\# \to \mathrm{Ext}^{3,2n+1}_{\mathscr{A}}(M)$$
$$\to (M/2)^\# \overset{\psi^3-1}{\to} (M/2)^\# \to \cdots ,$$

which yields the case n odd of Theorem 3.1 after dualization. The case n even is similar.

For the splitting of the short exact sequences of Theorem 3.1, we use an h_1-action on the Ext groups, as described in the following proposition, which will be proved at the end of this section.

Proposition 3.6. *There is a Yoneda (composition) product in* $\mathrm{Ext}_{\mathscr{A}}$ *and an element*

$$h_1 \in \mathrm{Ext}^1_{\mathscr{A}}(QK^1(S^{2n+1})^\wedge, QK^1(S^{2n+3})^\wedge)$$

satisfying

(1) $2h_1 = 0$;
(2) *Yoneda product with* h_1 *corresponds to the* h_1*-action in the* BTSS *under the isomorphism of Theorem* 1.1;
(3) *Under the short exact sequences of Theorem* 3.1 *with* $s+n$ *odd, there is a commutative diagram of short exact sequences*

$$\begin{array}{ccccc} \operatorname{coker}(\theta|M/2) & \longrightarrow & \operatorname{Ext}_{A}^{s+1,2n+3}(M)^{\#} & \longrightarrow & \ker(\theta|M_2) \\ {\scriptstyle 1}\downarrow & & {\scriptstyle h_1^{\#}}\downarrow & & {\scriptstyle 1}\downarrow \\ \operatorname{coker}(\theta|M/2) & \longrightarrow & \operatorname{Ext}_{A}^{s,2n+1}(M)^{\#} & \longrightarrow & \ker(\theta|M_2) \end{array}$$

and a similar one when $s+n$ *is even.*

The splitting in Theorem 3.1 follows now, since the five lemma, applied to the dual of the diagram of Proposition 3.6(3), implies that $\cdot h_1$ is an isomorphism on $\operatorname{Ext}_{\mathscr{A}}^{s,2n+1}(M)$, and so, since $2h_1=0$, $\operatorname{Ext}_{\mathscr{A}}^{s,2n+1}(M)$ can have no elements of order 4. $\square$

Proposition 3.6 shows that the h_1-action is an isomorphism from $\operatorname{Ext}_{\mathscr{A}}^{s,2n+1}(M)$ to $\operatorname{Ext}_{\mathscr{A}}^{s+1,2n+3}(M)$ for $s \geqslant 2$. When $s=1$ and n is even, the diagram of Proposition 3.6(3) applies to describe the action of h_1, which is injective. When $s=1$ and n is odd, the situation is a bit more delicate. The following result is proved in [9]. We forego giving the proof here because it is quite involved, and the result plays only a minor role in our applications in Section 4.

Proposition 3.7. *Suppose* $\psi^{-1}=-1$ *in* $QK^1(X)$. *Let*

$$N=(QK^1(X)/\operatorname{im}(\psi^2))^{\#},$$

$N_2=\ker(2|N)$, *and* $\theta=\psi^3-3^n$ *with* n *odd. There is a commutative diagram of exact sequences*

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & E_2^{1,2n+1}(X) & \xrightarrow{i} & N & \xrightarrow{\theta^{\#}} & N \\ & & \downarrow & & {\scriptstyle h_1\cdot}\downarrow & & {\scriptstyle \rho_2}\downarrow & & {\scriptstyle \rho_2}\downarrow \\ N_2 & \xrightarrow{\theta^{\#}} & N_2 & \xrightarrow{j} & E_2^{2,2n+3}(X) & \longrightarrow & N/2 & \xrightarrow{\theta^{\#}} & N/2 \end{array}$$

If $\rho_2(i(x))=0$, *then* $h_1x=j(\theta^{\#}(x/2))$, *which is well-defined as an element of* $\operatorname{coker}(\theta^{\#}|N_2)$.

Corollary 3.8. *With the hypotheses of Proposition* 3.7, $h_1\cdot$ *induces a monomorphism*

$$E_2^{1,2n+1}(X)/2 \to E_2^{2,2n+3}(X)/2.$$

Proof. Suppose $h_1x=0$. Then $\theta^{\#}(i(x))=0$ and $i(x)=2y$ for some $y\in N$. By Proposition 3.7, $h_1x=j(\theta^{\#}(y))$ considered as an element of $\operatorname{coker}(\theta^{\#}|N_2)$. Since h_1x is assumed to be 0, we deduce $\theta^{\#}(y)=\theta^{\#}(z)$ with $2z=0$. Then $y-z\in\ker(\theta^{\#})$. Hence there is an element $y'\in E_2^{1,2n+1}(X)$ satisfying $i(y')=y-z$ and $2y'=x$. Hence $x=0\in E_2^{1,2n+1}(X)/2$. $\square$

The following proposition was used earlier in this section.

Proposition 3.9. *If* M *is a finite object of* Inv, *then* $\operatorname{Ext}_{Inv}^{s}(\mathbb{Q}^{(\varepsilon)},M)=0$ *for* $s\geqslant 0$.

Proof. The object M must be isomorphic to a sum of $(\mathbb{Z}/2^n)^{(\varepsilon')}$'s plus copies of $P/2^n=\mathbb{Z}/2^n\oplus\mathbb{Z}/2^n$ with ψ^{-1} interchanging factors.

By Bousfield [14, 3.10], $\mathrm{Ext}^s_{Inv}(\mathbb{Q}^{(\varepsilon)}, M) = 0$ for $s > 1$. For $s = 0$, we have

$$\mathrm{Hom}_{Inv}(\mathbb{Q}^{(\varepsilon)}, M) \subset \mathrm{Hom}_{\mathrm{AbGp}}(\mathbb{Q}, M) = 0.$$

Let $0 \to R \to F \to \mathbb{Q} \to 0$ be a projective resolution in AbGp. Then $0 \to R^{(\varepsilon)} \to F^{(\varepsilon)} \to \mathbb{Q}^{(\varepsilon)} \to 0$ is a projective resolution in Inv by Bousfield [14, 3.6]. We first consider the case $\varepsilon' = \varepsilon$. Here $\mathrm{Hom}_{Inv}(F^{(\varepsilon)}, (\mathbb{Z}/2^n)^{(\varepsilon)}) = \mathrm{Hom}_{\mathrm{AbGp}}(F, \mathbb{Z}/2^n)$, and so $\mathrm{Ext}^1_{Inv}(\mathbb{Q}^{(\varepsilon)}, (\mathbb{Z}/2^n)^{(\varepsilon)}) = \mathrm{Ext}_{\mathrm{AbGp}}(\mathbb{Q}, \mathbb{Z}/2^n)$. Using the injective resolution in AbGp

$$0 \to \mathbb{Z}/2^n \to \mathbb{Q}/\mathbb{Z} \xrightarrow{2^n} \mathbb{Q}/\mathbb{Z} \to 0,$$

one readily verifies $\mathrm{Ext}_{\mathrm{AbGp}}(\mathbb{Q}, \mathbb{Z}/2^n) = 0$.

With $\varepsilon' = -\varepsilon$, we have $\mathrm{Hom}_{Inv}(F^{(\varepsilon)}, (\mathbb{Z}/2^n)^{(-\varepsilon)}) = \mathrm{Hom}_{\mathrm{AbGp}}(F, \mathbb{Z}/2)$. Arguing as above with $n=1$, we obtain $\mathrm{Ext}^1_{Inv}(\mathbb{Q}^{(\varepsilon)}, (\mathbb{Z}/2^n)^{(-\varepsilon)}) = 0$. Finally, $\mathrm{Ext}^1_{Inv}(\mathbb{Q}^{(\varepsilon)}, P/2^n) = 0$ follows from $\mathrm{Hom}_{Inv}(F^{(\varepsilon)}, P/2^n) = 0$ by a similar argument. □

The generalization of Theorem 3.1 and Proposition 3.2 to an arbitrary M is given by the following result, whose proof is a straightforward generalization of methods used above.

Theorem 3.10. (a) *Let $p = 2$ and let M be a finite stable p-adic Adams module. Let $\theta_n = \psi^3 - 3^n$, and*

$$Q_m = \frac{\ker(1 - (-1)^m \psi^{-1})}{\mathrm{im}(1 + (-1)^m \psi^{-1})}.$$

Then

- $\mathrm{Ext}^{1,2n+1}_{\mathscr{A}}(M)^{\#} \approx \mathrm{coker}(\theta_n | \mathrm{coker}(1 - (-1)^n \psi^{-1}))$;
- *there is a short exact sequence*

$$0 \to \mathrm{coker}(\theta_n | Q_n) \to \mathrm{Ext}^{2,2n+1}_{\mathscr{A}}(M)^{\#} \to \ker(\theta_n | \mathrm{coker}(1 - (-1)^n \psi^{-1})) \to 0;$$

- *for $s > 2$, there is a short exact sequence*

$$0 \to \mathrm{coker}(\theta_n | Q_{s+n}) \to \mathrm{Ext}^{s,2n+1}_{\mathscr{A}}(M)^{\#} \to \ker(\theta_n | Q_{s+n-1}) \to 0.$$

(b) *If M is a 2-local abelian group with involution ψ^{-1}, then*

$$\mathrm{Ext}^s_{Inv}(\mathbb{Z}_{(2)}^{((-1)^n)}, M) \approx \begin{cases} Q_{s+n} & \text{if } s > 0, \\ \ker(1 - (-1)^n \psi^{-1}) & \text{if } s = 0. \end{cases}$$

We complete this section by proving Proposition 3.6.

Proof of Proposition 3.6. We apply Proposition 3.3 and (3.4) to obtain an exact sequence

$$\mathrm{Hom}_{Inv}((\mathbb{Q}/\mathbb{Z})^{((-1)^{n+1})}, (\mathbb{Q}/\mathbb{Z})^{((-1)^n)}) \to \mathrm{Ext}^1_{\mathscr{A}}(QK^1(S^{2n+1})^\wedge, QK^1(S^{2n+3})^\wedge)$$
$$\to \mathrm{Ext}^1_{Inv}((\mathbb{Q}/\mathbb{Z})^{((-1)^{n+1})}, (\mathbb{Q}/\mathbb{Z})^{((-1)^n)}) \xrightarrow{2\cdot\mathrm{odd}}.$$

Using (3.5), and then arguing as in the proof of Proposition 3.9, we obtain

$$\mathrm{Ext}^1_{Inv}((\mathbb{Q}/\mathbb{Z})^{((-1)^{n+1})}, (\mathbb{Q}/\mathbb{Z})^{((-1)^n)}) \approx \mathrm{Ext}^0_{Inv}(\mathbb{Z}_{(2)}^{((-1)^{n+1})}, (\mathbb{Q}/\mathbb{Z})^{((-1)^n)})$$

and similarly to Proposition 3.2, this is $\mathbb{Z}/2$, generated by $\frac{1}{2} \in \mathbb{Q}/\mathbb{Z}$ on the RHS. The nonzero element is called h_1. Since $h_1 \in \mathbb{Z}/2$, $2h_1 = 0$. Part 2 of the proposition follows since the isomorphism of Theorem 1.1 respects Yoneda products, and the two notions of h_1 must agree since they are the only nonzero element in isomorphic groups.

For part 3, we first consider the Yoneda product in Ext_{Inv}

$$\mathrm{Ext}^s_{Inv}(\mathbb{Z}^{(\varepsilon)}_{(2)}, M) \otimes \mathrm{Ext}^1_{Inv}(\mathbb{Z}^{(-\varepsilon)}_{(2)}, \mathbb{Z}^{(\varepsilon)}_{(2)}) \to \mathrm{Ext}^{s+1}_{Inv}(\mathbb{Z}^{(-\varepsilon)}_{(2)}, M).$$

With P as in the proof of Proposition 3.2, composition with h_1 is defined by the diagram

$$\begin{tikzcd}
0 & \mathbb{Z}^{(-\varepsilon)}_{(2)} \arrow[l] & P \arrow[l] & P \arrow[l, "1+\varepsilon\psi^{-1}"'] \arrow[dl] \arrow[d, "1"] & P \arrow[l, "1-\varepsilon\psi^{-1}"'] \arrow[d, "1"] & \cdots \arrow[l] & P \arrow[d, "1"] \\
& & \mathbb{Z}^{(\varepsilon)}_{(2)} & P \arrow[l] & P \arrow[l, "1-\varepsilon\psi^{-1}"'] & \cdots \arrow[l] & P \arrow[d] \\
& & & & & & M
\end{tikzcd}$$

Since the chain map of resolutions can be chosen to be the identity, the composition is the identity under the identifications given in Proposition 3.2. Part 3 of the proposition follows since the morphisms of Proposition 3.3 and (3.4) are compatible with Yoneda products. □

4. The BTSS of F_4

In this section, we prove Theorem 1.2. There are three steps.

(1) Use Theorems 1.1 and 3.1 to compute the E_2-term of the BTSS converging to $v_1^{-1}\pi_*(\hat{F_4})$.
(2) Use the fibration

$$G_2 \to F_4 \to F_4/G_2 \tag{4.1}$$

to determine the differentials and extensions in the spectral sequence.
(3) Show that $F_4 \to \hat{F_4}$ induces an isomorphism in $v_1^{-1}\pi_*(-)$. This was stated in Theorem 1.4 and is proved in Section 5.

From [19, 3.8] we have

Proposition 4.2. *There is a basis $\{v_1, v_2, v_3, v_4\}$ of $QK^1(F_4)$ on which $\psi^{-1} = -1$ and the transposes of the matrices of ψ^2 and ψ^3 are given by*

$$(\psi^2)^{\mathrm{T}} = \begin{pmatrix} 2 & 3 & 1 & 0 \\ 0 & 32 & -8 & -1 \\ 0 & 0 & 128 & -24 \\ 0 & 0 & 0 & 2048 \end{pmatrix} \quad \textit{and} \quad (\psi^3)^{\mathrm{T}} = \begin{pmatrix} 3 & 24 & 15 & -1 \\ 0 & 3^5 & -162 & -81 \\ 0 & 0 & 3^7 & -3^7 \\ 0 & 0 & 0 & 3^{11} \end{pmatrix}.$$

This can be shown to agree with the Chern character calculation of [26, 4.8].

By Theorems 1.1 and 3.1, $E_2^{1,4k+3}(F_4)^{\#}$ is obtained from the following result.

Proposition 4.3. *If* $(\psi^2)^{\mathrm{T}}$ *and* $(\psi^3)^{\mathrm{T}}$ *are as in Proposition* 4.2, *then the abelian group presented by the matrix*

$$\begin{pmatrix} (\psi^2)^{\mathrm{T}} \\ (\psi^3 - 3^{2k+1})^{\mathrm{T}} \end{pmatrix}$$

is $\mathbb{Z}/2^{\min(12,6+2\nu(k-5))}$.

Proof. Replace 3^{2k+1} by $3^{10}(R+3)$ in the matrix. Then $\nu(R)=\nu(k-5)+3$. Pivot the matrix on the entries in position $(1,3)$, then $(2,4)$, and then $(5,2)$, which will have become odd. This leaves five relations on the first generator, which, up to odd multiples, are

$$2^{14}+2^6R,$$

$$2^{18}+2^{14}R,$$

$$2^{12}+2^8R+R^2,$$

$$2^{12}+2^6R+R^2,$$

$$2^7R+2^3R^2.$$

Then the exponent of 2 in the fourth relation is $\min(12,2\nu(R))$, and all other relations are at least that 2-divisible. □

The order of $\ker((\psi^3-3^{2k+1})|QK^1(F_4)/\mathrm{im}(\psi^2))$, which is a summand of $E_2^{2,4k+3}(F_4)^{\#}$, equals that of the cokernel, which was determined in the preceding proposition. For the group structure, we need

Proposition 4.4. *The group* $\ker((\psi^3-3^{2k+1})|QK^1(F_4)/\mathrm{im}(\psi^2))$ *is cyclic.*

Proof. Let $M=QK^1(F_4)/\mathrm{im}(\psi^2)$, $M_2=\ker(\cdot 2|M)$, and $K=\ker((\psi^3-3^{2k+1})|M)$. The number of summands in K equals the dimension of $M_2\cap K$. Note that $\psi^3-3^{2k+1}=\psi^3-1$ on M_2.

A basis for M_2 is given by $\psi^2(v_3)/2$ and $\psi^2(v_4)/2$. We have

$$(\psi^3-1)(\psi^2(v_4)/2)=\psi^2(\psi^3-1)(v_4)/2=\psi^2(\tfrac{3^{11}-1}{2}\,v_4)\equiv 0\in M$$

and

$$(\psi^3-1)(\psi^2(v_3)/2)=\psi^2(\tfrac{3^7-1}{2}\,v_3-\tfrac{3^7}{2}\,v_4)\equiv\psi^2(v_4/2)\in M.$$

Thus $M_2\cap K=\langle\psi^2(v_4)/2\rangle$ is one-dimensional. □

For the elements of higher filtration in $E_2(F_4)$, we need

Proposition 4.5. *Let* $M=QK^1(F_4)/\mathrm{im}(\psi^2)$. *Then* $(\psi^3-1)|(M/2)$ *has kernel* $\approx\mathbb{Z}/2$ *with basis* $\{v_2\sim v_3\}$ *and cokernel* $\approx\mathbb{Z}/2$ *with basis* $\{v_1\}$, *while* $(\psi^3-1)|M_2$ *has kernel* $\approx\mathbb{Z}/2$ *with basis* $\{\psi^2(v_4)/2\}$ *and cokernel* $\approx\mathbb{Z}/2$ *with basis* $\{\psi^2(v_3)/2\}$.

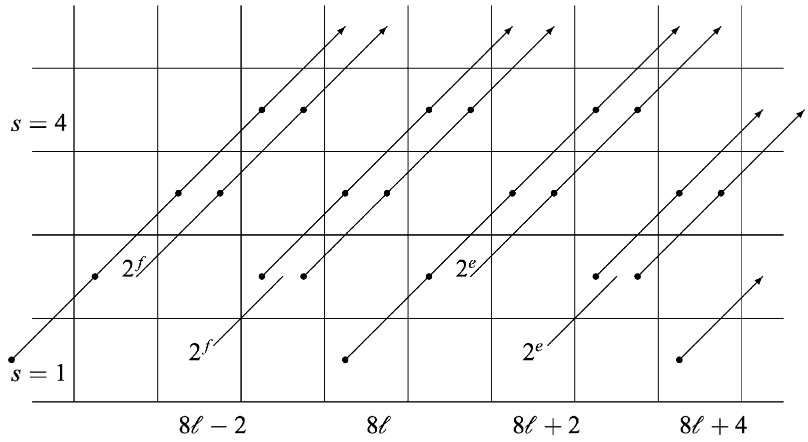

Fig. 1.

Proof. We have $M/2 \approx \mathbb{Z}/2 \oplus \mathbb{Z}/2$ with basis $\{v_1, v_2 \sim v_3\}$. We have that $(\psi^3-1)v_1 \equiv v_3$ and $(\psi^3-1)v_2 \equiv 0 \bmod (2, \mathrm{im}(\psi^2))$. In the proof of Proposition 4.4, ψ^3-1 on M_2 was analyzed. □

We obtain the following diagram of $E_2(F_4)$ with $e=6$ and $f=\min(12, 8+2\nu(\ell-3))$.

Here, as usual with Adams spectral sequence types of diagrams, the horizontal grading is $t-s$, and classes in $E_\infty^{*,*+i}(X)$ provide an associated graded for $\pi_i(\hat{X})$. Each dot represents $\mathbb{Z}/2$, and an integer represents a cyclic summand of that order. The diagonal lines indicate multiplication by h_1 in the BTSS (3.6), which corresponds to the Hopf map η in homotopy. We call these "η-towers". The action of h_1 on the 1-line can be computed explicitly using Proposition 3.7; see Proposition 4.15. For most of our purposes, we just need that it is nonzero on the 1-line (Corollary 3.8), and so we do not indicate explicitly which element it hits in Fig. 1.

Because $\eta^4 = 0$ in $\pi_{n+4}(S^n)$, there must be a pattern of d_3-differentials which annihilates all η-towers in large filtration. However, careful consideration is required to determine whether a particular η-tower supports a d_3-differential or is hit by one. This will affect whether or not a few elements at the bottom of the η-tower survive the spectral sequence.

In order to determine the d_3-differentials in F_4, we use fibration (4.1). The groups $v_1^{-1}\pi_*(G_2;2)$ were computed in [23] using homotopy theoretic methods. We now show how these groups can be seen in the BTSS.

From [19, 3.7], $QK^1(G_2)$ has a basis $\{g_1, g_2\}$ on which $(\psi^2)^{\mathrm{T}}$ and $(\psi^3)^{\mathrm{T}}$ are given by

$$(\psi^2)^{\mathrm{T}} = \begin{pmatrix} 2 & -15 \\ 0 & 32 \end{pmatrix} \quad \text{and} \quad (\psi^3)^{\mathrm{T}} = \begin{pmatrix} 3 & -120 \\ 0 & 3^5 \end{pmatrix}.$$

By methods similar to those employed above for F_4, we obtain

Proposition 4.7. *Let* $M' = QK^1(G_2)/\mathrm{im}(\psi^2)$.

(1) $E_2^{1,4k+3}(G_2)^{\#} \approx \mathbb{Z}/2^{\min(6,\nu(k-2)+3)} \approx \ker((\psi^3 - 3^{2k+1})|M')$;

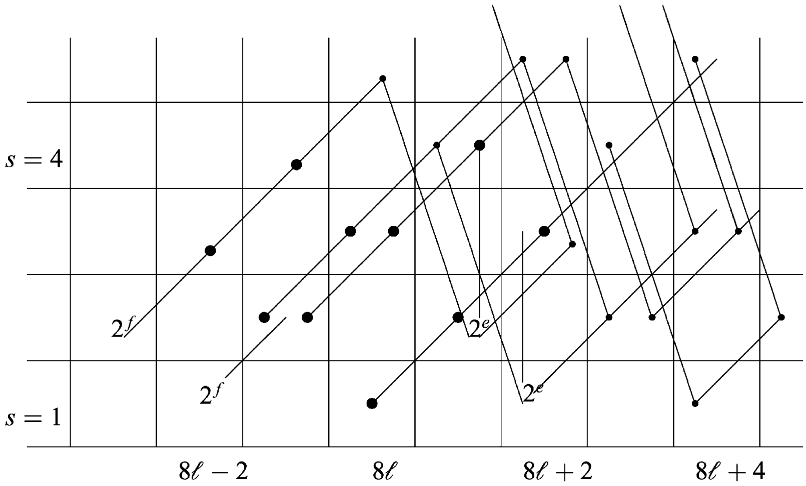

Fig. 2.

(2) *$M'/2 \approx \mathbb{Z}/2$, generated by g_1, with $\psi^3-1=0$ on $M'/2$, and $M_2'=\mathbb{Z}/2$, generated by $16g_2$, with $\psi^3-1=0$ on M_2'.*

Thus by Theorems 1.1 and 3.1 $E_2(G_2)$ has the form of Fig. 1, with $e=\min(6,v(\ell-1)+4)$ and $f=3$.

The following result will be proved in Section 5, simultaneously with the proof that Theorem 1.4 holds for G_2. The proof utilizes the map $G_2\to S^6$ with fiber $SU(3)$, the analysis of $v_1^{-1}\pi_*(G_2)$ in [23], and the fact that S^6 satisfies the CTP.

Theorem 4.8. *The differentials and extensions in the* BTSS *of G_2 are as in* Fig. 2, *with $e=\min(6,v(\ell-1)+4)$ and $f=3$.*

Note that this diagram for $\ell-1$ would provide additional η-towers which are not displayed on the left side of the diagram.

We need also the BTSS and v_1-periodic homotopy groups of F_4/G_2. We use [23, 1.1], which states that there is a 2-local fibration

$$S^{15}\to F_4/G_2\to S^{23}. \tag{4.10}$$

Proposition 4.11. *F_4/G_2 satisfies the* CTP.

Proof. In Definition 5.3, we define the notion of a K_*-strongly spherically resolved space, and in Proposition 5.5 we prove that such spaces satisfy the CTP. This notion is slightly stronger than previous notions of spherical resolvability, in that we require that each $K_*(X_i)$ is isomorphic to an exterior algebra as a K_*K-coalgebra. Since the X_i will not usually have a product structure, it will not be automatic that the coproduct and coaction on classes that appear to be products of primitive elements satisfy the requisite formulas. However, for odd-sphere bundles over odd spheres, this is not a problem.

In the case at hand, $X = F_4/G_2$, using $\mathbb{Z}/2$-graded K-homology, $K_1(X)$ has basis consisting of two primitive classes, x_1 coming from S^{15}, and x_2 mapping to S^{23}, while $K_0(X)$ has basis consisting of a single class x_3 whose coproduct involves $x_1 \otimes x_2 + x_2 \otimes x_1$, as can be seen by duality from the algebra $K^*(X)$. The K_*K-coaction must satisfy $\psi(x_1) = 1 \otimes x_1$ by naturality, $\psi(x_3) = 1 \otimes x_3$ since $(K_*K)_{\mathrm{od}} = 0$, and $\psi(x_2) = 1 \otimes x_2 + h \otimes x_1$ for some $h \in K_*K$. Since $x_1^2 = 0$ in an exterior algebra, these formulas for ψ are consistent with an exterior algebra in which $x_3 = x_1 x_2$, i.e., $\psi(x_3) = \psi(x_1)\psi(x_2)$. □

Since the attaching map in F_4/G_2 is σ, by Adams [1, 7.5, 7.17], we have

Proposition 4.12. *$QK^1(F_4/G_2)$ has basis $\{w_1, w_2\}$ with $\psi^k w_2 = k^{11} w_2$ and*

$$\psi^k w_1 = k^7 w_1 + \frac{uk^7(k^4 - 1)}{16} w_2$$

with u odd.

Applying to this the methods of Propositions 4.3–4.5 and 4.7, we obtain that the BTSS-E_2 for F_4/G_2 has the form of Fig. 1 with $e = 6$ and $f = \min(12, 7 + \nu(\ell - 19))$.

Theorem 4.13. *The differentials and extensions in the* BTSS *of F_4/G_2 are as in Fig.* 2 *with $e = 6$ and $f = \min(12, 7 + \nu(\ell - 19))$.*

Proof. We need the fact [6, p. 488]; [3, p. 352] that the BTSS of S^{8m-1} has the form of Fig. 2 with $e = 3$ and $f = \min(4m - 1, \nu(\ell - m) + 4)$. Fibration (4.10) induces a short exact sequence in $QK^1(-)$, a long exact sequence in E_2 of the BTSS, and a long exact sequence in $v_1^{-1}\pi_*(-)$. The d_3-differentials on the η-towers emanating from $(t - s, s) = (8\ell + 3, 2)$ in S^{15} and S^{23} force similar d_3-differentials in F_4/G_2, as do the d_3-differentials on the η-towers emanating from $(8\ell + 4, 1)$. Since $E_2^{2,8\ell+3}(F_4/G_2) \to E_2^{2,8\ell+3}(S^{23})$ maps onto the $\mathbb{Z}/8$, which supports a d_3-differential, d_3 must be nonzero on $\mathbb{Z}/2^6 \subset E_2^{2,8\ell+3}(F_4/G_2)$ and on the η-tower arising from it.

For $s = 1$ and 2, $E_2^{s,8\ell+3}(S^{15}) \to E_2^{s,8\ell+3}(F_4/G_2)$ is a monomorphism $\mathbb{Z}/8 \hookrightarrow \mathbb{Z}/2^6$; thus the nonzero extension $(\cdot 2)$ from $E_2^{s,8\ell+3}(F_4/G_2)$ to $E_2^{s+2,8\ell+5}(F_4/G_2)$ will follow from that in S^{15} once we see that the η-tower into which $E_2^{s,8\ell+3}(S^{15})$ extends maps across. When $s = 1$, this is clear, since the extension is into $h_1^2 E_2^{1,8\ell+1}$, and $E_2^1(S^{15}) \to E_2^1(F_4/G_2)$ must be injective.

The case s=2 requires more care. For $s \geqslant 2$, the two classes of $E_2^{s,8\ell-3+2s}(S^{15})$ can be characterized as "stable" and "unstable". From the point of view of Theorem 3.1, the one in $\ker(\theta|M/2)$ is stable, while the one in $\mathrm{coker}(\theta|M_2)$ is unstable. This is true because elements of M_2, being $\psi^2 x/2$, depend on the dimension of the sphere, while elements in $M/2$ are generators of M and independent of the dimension of the sphere. The extension in BTSS(S^{15}) in $t - s = 8\ell + 1$ is into the unstable class. This is true because the large summand in $E_2^2(S^{2n+1})$ is unstable.

Now consider the commutative diagram of exact sequences, with $\bar{K}(X) = QK^1(X)/\mathrm{im}(\psi^2)$ and $\mathrm{Ext} = \mathrm{Ext}^{s,8\ell-3+2s}$,

$$\begin{array}{ccccccccc} \xrightarrow{\theta} & \overline{K}(F_4/G_2)_2 & \longrightarrow & \mathrm{Ext}_{\mathcal{A}}(\overline{K}(F_4/G_2))^{\#} & \longrightarrow & \overline{K}(F_4/G_2)/2 & \xrightarrow{\theta} \\ & \downarrow & & \downarrow & & \downarrow & \\ \xrightarrow{\theta} & \overline{K}(S^{15})_2 & \longrightarrow & \mathrm{Ext}_{\mathcal{A}}(\overline{K}(S^{15}))^{\#} & \longrightarrow & \overline{K}(S^{15})/2 & \xrightarrow{\theta} \end{array}$$

With w_i as in Proposition 4.12,

$$(\psi^3 - 1)(\psi^2 w_1/2) = \psi^2(\tfrac{3^7-1}{2}\, w_1 + \tfrac{3^7(3^4-1)}{2}\, w_2) \equiv \psi^2(\tfrac{w_2}{2}) \bmod \mathrm{im}\, \psi^2$$

and

$$(\psi^3 - 1)(\psi^2 w_2/2) = \psi^2(\tfrac{3^{11}-1}{2}\, w_2) \equiv 0 \bmod \mathrm{im}\, \psi^2.$$

Thus $\mathrm{coker}(\theta|\bar{K}(F_4/G_2)_2) = \langle \psi^2 w_1/2 \rangle$, whose image in $\mathrm{coker}(\theta|\bar{K}(S^{15})_2)$ is nontrivial. Dually, the unstable class in $\mathrm{Ext}_{\mathcal{A}}^{s,8\ell-3+2s}(\bar{K}(S^{15}))$ maps nontrivially to F_4/G_2, and hence the extension from $E_\infty^{2,8\ell+3}(F_4/G_2)$ to $E_\infty^{4,8\ell+5}(F_4/G_2)$ is nontrivial.

We must check that the element into which we have just seen the nontrivial extension in $v_1^{-1}\pi_{8\ell+1}(F_4/G_2)$ hits is not killed by a d_3-differential. It looks that way in Fig. 2, but we have not been precise here in depicting which element is hit by h_1 from position $(8\ell + 2, 1)$, which affects which element is hit by d_3 from $(8\ell + 2, 1)$. In Proposition 4.15, we will see that, using $\{w_2, \frac{1}{2}\,\psi^2 w_1\}$ as basis for the eta towers in $E_2^{s,2n+1}(F_4/G_2)$ with $s+n$ even, h_1 from $E_2^{1,8\ell+3}$ hits the sum of the basis elements and hence so does d_3, since $d_3 h_1 = h_1 d_3$. Thus the extension is into a nonzero homotopy class. □

The result for $v_1^{-1}\pi_*(F_4/G_2)$ that can be read off from Theorem 4.13 differs slightly from [20, 8.10]. A mistake in [20] was discussed in [8]. The key lemma [20, 8.16] is false, and this caused the evaluation of a d_6-differential in [20, p. 1045] to be incorrect.

Now we can prove the following result, from which Theorem 1.2 follows immediately, once we know Theorem 1.4 for F_4.

Theorem 4.14. *The differentials and extensions in the* BTSS *of F_4 are as in Fig.* 2 *with $e = 6$ and $f = \min(12, 8 + 2\nu(\ell - 3))$.*

Proof. Fibration (4.1) induces a short exact sequence in $QK^1(-)$ and a long exact sequence in E_2. The d_3-differentials on the η-towers emanating from $(t-s,s)=(8\ell+3,2)$ and $(8\ell+4,1)$ are implied by their existence in G_2 and F_4/G_2. The nonzero d_3 from $(8\ell + 2, 1)$ follows by Corollary 3.8.

The $\mathbb{Z}/2^6$ summand in $E_2^{2,8\ell+3}$ maps isomorphically from F_4 to F_4/G_2, as does the η-tower arising from it. Thus the d_3 on this η-tower in F_4/G_2 implies the same in F_4, and the nontrivial extension from $E_2^{2,8\ell+3}(F_4/G_2)$ implies the same in F_4.

Finally, both $E_2^{1,8\ell+3}(G_2) \to E_2^{1,8\ell+3}(F_4)$ and $E_2^{1,8\ell+1}(G_2) \to E_2^{1,8\ell+1}(F_4)$ are injective, and so the extension in G_2 from $E_\infty^{1,8\ell+3}$ to $h_1^2 \cdot E_2^{1,8\ell+1}$ implies the same in F_4. □

Although it seems to be not necessary for our conclusions, except perhaps as it was used at the end of the proof of Theorem 4.13, we list the result of applying Proposition 3.7 to obtain the

h_1-action on the 1-line of the spaces we are studying here. It is slightly easier to state and prove if we dualize the diagram of Proposition 3.7.

Proposition 4.15. *Let $X = G_2$, F_4/G_2, or F_4, and $M = QK^1(X)/\mathrm{im}(\psi^2)$. Then $E_2^{2,4k+5}(X)^\#$ has basis with one nonzero element from* $\ker(\theta|M/2)$, *which we call "stable", and one nonzero element from* $\mathrm{coker}(\theta|M_2)$, *which we call "unstable". In the notation used earlier in this section, the unstable elements are, respectively, $16g_2$, $\frac{1}{2}\psi^2 w_1$, and $\frac{1}{2}\psi^2 v_3$, while the stable elements are, respectively, g_1, w_2, and v_2. For G_2, $h_1^\#$ is nonzero on the stable class iff $v(k-2) \leqslant 3$, and is nonzero on the unstable class iff $v(k-2) \geqslant 3$. For F_4/G_2, $h_1^\#$ is nonzero on the stable class iff $v(k-5) = 0$ or 5, and is nonzero on the unstable class except perhaps when $v(k-5) = 5$. For F_4, $h_1^\#$ is nonzero on the stable class iff $v(k-5) \neq 3$ and is nonzero on the unstable class iff $v(k-5) = 3$.*

Proof. We illustrate with the proof for F_4. The others are similar and slightly easier.

The dual of the diagram of Proposition 3.7 is

$$\begin{tikzcd} M_2 \arrow[r, "\theta"] \arrow[d, "\imath"'] & M_2 \arrow[r] \arrow[d, "\imath"'] & E_2^{2,4k+5}(X)^\# \arrow[r] \arrow[d, "h_1^\#"'] & M/2 \arrow[r, "\theta"] \arrow[d] & M/2 \\ M \arrow[r, "\theta"] & M \arrow[r] & E_2^{1,4k+3}(X)^\# \arrow[r] & 0 & \end{tikzcd}$$

An unstable element is the image in $E_2^{2,4k+5}(X)^\#$ of some $\frac{1}{2}\psi^2 x \in M_2$. It will have a nonzero image under $h_1^\#$ iff adjoining the relation $\frac{1}{2}\psi^2 x$ to the presentation matrix of Proposition 4.3 decreases the order of the group. A stable element is the pullback to $E_2^\#$ of an element $y \in M/2$ for which $\theta(y) = 2z$. By Proposition 3.7, this will have a nonzero image under $h_1^\#$ iff adjoining the relation z $(= \frac{1}{2}\theta(y))$ decreases the order of the group.

If $\frac{1}{2}\psi^2 v_3$ is added to the presentation matrix for F_4, this has the effect of dividing the first of the five relations listed in the proof of Proposition 4.3 by 2, yielding $2^{13} + 2^5 R$. This lowers the order of the group only when $v(R) = 6$.

Note that $\theta(v_2)$ is divisible by 2 in M because $\theta(v_2) - \psi^2(v_2)$ is divisible by 2 in $QK^1(F_4)$. We add $\frac{1}{2}(\theta(v_2) - \psi^2(v_2))$ to the presentation matrix, which has the effect of dividing the third of the five relations by 2, yielding $2^{11} + 2^7 R + \frac{1}{2}R^2$. This lowers the order of the group unless $v(R) = 6$. □

5. The completion telescope property

In this section, we first show that the BTSS converges to $v_1^{-1}\pi_*(\hat{X})$ for a class of spaces X which includes spheres and simply connected finite H-spaces. Then we prove Theorem 1.4, which is the isomorphism $v_1^{-1}\pi_*(X) \approx v_1^{-1}\pi_*(\hat{X})$ for certain important spaces X.

We begin by recalling some of the results of [10]. For a space X, the K-completion of X, denoted $\hat{X}$, is constructed as Tot of a cosimplicial space constructed from the K-theory spectrum. There is a natural transformation $X \to \hat{X}$. The Bousfield–Kan spectral sequence associated to the standard filtration of the K-theory Tot is the BTSS. We are using here the v_1-periodic BTSS, although the difference between this and the unlocalized BTSS is inconsequential for our purposes here, since they agree in sufficiently large dimensions.

Proposition 5.1 (Bendersky and Thompson [10, 2.3]). *Suppose that there is an N and r such that $E_r^{s,t}(X)=0$ if $s>N$. Then the* BTSS *of X converges to the v_1-periodic homotopy groups of $\hat{X}$.*

Proof. It is shown in [10, 2.3] that, under this hypothesis, the unlocalized BTSS converges to $\pi_*(\hat{X})$. Because there is a horizontal vanishing line, the v_1-periodic BTSS converges to $v_1^{-1}\pi_*(\hat{X})$, since there can be no v_1-periodic family of classes or differentials of increasingly large filtration. □

One family of spaces for which we can prove there is a horizontal vanishing line is the algebraically spherically resolved spaces [11].

Definition 5.2. A space X is *K_*-algebraically spherically resolved* (ASR), if:

(1) $K_*(X)$ is a free K_*-module.
(2) There is a K_*K-subcomodule $M\subset K_{\mathrm{od}}(X)$ such that $K_*(X)\approx\Lambda(M)$ as K_*K-comodules.
(3) If $\Lambda(M)$ is made into a coalgebra by making M primitive, then the isomorphism is as $K_*(K)$-coalgebras.
(4) One can choose a basis $\{m_1,m_2,\ldots\}$ for M so that each sequence

$$0\to K_*\{m_1,\ldots,m_{n-1}\}\to K_*\{m_1,\ldots,m_n\}\to K_*\{m_n\}\to 0$$

is a short exact sequence of $K_*(K)$-comodules.

A space is ASR if from the point of view of K-theory it appears as if it is built out of a finite sequence of fibrations over odd spheres. The geometric analogue is given in the following definition.

Definition 5.3. A space X is *K_*-strongly spherically resolved* (SSR) if there are spaces $*=X_0$, $X_1,\ldots,X_k=X$ and fibrations

$$X_{i-1}\to X_i\to S^{n_i} \tag{5.4}$$

with n_i odd such that the K-homology groups of (5.4) form a split extension

$$\Lambda(x_1,\ldots,x_{i-1})\to\Lambda(x_1,\ldots,x_i)\to\Lambda(x_i)$$

as K_*K-coalgebras.

Proposition 5.5. *Suppose X is either K_*-SSR or a simply-connected finite H-space with $H_*(X;\mathbb{Q})$ associative. Then X is K_*-ASR and satisfies the hypotheses of Theorem 1.1.*

Proof. The proof when X is SSR is completely straightforward. If X is a simply-connected finite H-space with $H_*(X;\mathbb{Q})$ associative, then by Bousfield [15, 10.3, 10.4] $K_*(X;\mathbb{Z}_{(p)})$ is $\mathbb{Z}_{(p)}$-free and $K^*(X;\mathbb{Z}_p^\wedge)\approx\hat{\Lambda}(PK^1(X;\mathbb{Z}_p^\wedge))$. In [19, proof of 3.5], an algorithm was described for finding a basis of $QK^1(X)$ on which the matrix of the action of all Adams operations is triangular. We will sketch the algorithm in the next paragraph. Dualizing using [17, 2.3]; [13, Section 10], we obtain a basis of PK_1X on which the matrix of the K_*K-coaction is triangular. This provides the required filtration by exterior K_*K-subcoalgebras.

Sketch of algorithm: Since rationally X is a product of S^{2n_i+1}, there are elements x_i of $QK^1(X)$ on which $\psi^k(x_i)=k^{n_i}x_i$ for all k and such that rationally $\{x_i\}$ is a basis. Let M be the integer matrix expressing the x_i in terms of some initial basis B of the free abelian group $QK^1(X)$. If $\det(M)=\pm 1$, then $\{x_i\}$ is a basis of $QK^1(X)$ and has diagonal matrix for all ψ^k. If not, we will modify $\{x_i\}$ to remove divisors of $\det(M)$ and maintain the triangular property that, for all i and k, $\psi^k(x_i')$ involves only x_j' for $j \geqslant i$. Suppose $\{x_i'\}$ is such a set, with M' the matrix expressing it in terms of B, and p is a prime divisor of $\det(M')$. Then there is a nonzero vector $v=x_{i_0}'+\sum \alpha_j x_j'$ with all $j>i_0$ and $\alpha_j \in \mathbb{Z}$ such that v is divisible by p. Replace x_{i_0}' by $x_{i_0}''=\frac{1}{p}v$ in the set $\{x_i'\}$. The determinant of the matrix M'' expressing this new set in terms of B is $\det(M')/p$, and $\psi^k(x_{i_0}'')=k^{n_{i_0}}x_{i_0}''+$ a combination of x_j' with $j>i_0$, so the matrix of ψ^k stays triangular. □

The following result was proved in [11].

Proposition 5.6. *The v_1-periodic* BTSS *converges to $v_1^{-1}\pi_*(\hat{X})$ if X is K_*-algebraically spherically resolved.*

Proof. Briefly (for $p=2$), by the same argument as in [3, 5.4], the conditions guarantee that $E_2(X)$ is generated as an h_1-module by classes of filtration $\leqslant 2$. In a forthcoming paper of Bousfield [18], it will be shown that the Yoneda product in E_2 of the BTSS is associated to composition in homotopy. Since $\eta^4=0$ in homotopy, E_4 must have a horizontal vanishing line, and so Proposition 5.1 applies. □

There are other important examples of spaces which satisfy Proposition 5.1. Although ΩS^{2n+1} is not ASR, it has E_2 isomorphic to $E_2(S^{2n+1})$. Similarly, ΩS^{2n} is not ASR, but satisfies the condition of Proposition 5.1 because of Lemma 5.16.

We have established that the BTSS converges to $v_1^{-1}\pi_*(\hat{X})$ for many spaces. However our interest is in $v_1^{-1}\pi_*(X)$. Recall that X satisfies CTP (1.3) if $v_1^{-1}\pi_*(X)\to v_1^{-1}\pi_*(\hat{X})$ is an isomorphism. The rest of the paper is devoted to proving Theorem 1.4, which states that S^{2n}, ΩS^{2n}, G_2, and F_4 satisfy the CTP. The following lemma, which is a simple application of the five lemma, will be useful.

Lemma 5.7. *Suppose $X\to Y\to Z$ is a fibration for which $\hat{X}\to\hat{Y}\to\hat{Z}$ is also a fibration. If the* CTP *is true for any two of the spaces, then the third space satisfies the* CTP.

We now look for conditions that guarantee that a fibration satisfies Lemma 5.7. We observe that there is an easy solution to this problem at the odd primes. The K-homology of a finite simply connected H-space X is an exterior algebra generated by $PK_{\mathrm{od}}(X)\approx QK_{\mathrm{od}}(X)$. In this case $E_2(X)\approx \mathrm{Ext}_{\mathscr{V}}(K_*,QK_*(X))$, by 2.10 and 2.11. Moreover, at odd primes, by Bendersky and Thompson [11], $E_2^{s,2n+1}(X)\approx v_1^{-1}\pi_{2n+1-s}(\hat{X})$ for $s\in\{1,2\}$, and $E_2^s(X)=0$ if $s>2$. Thus if the fibration induces a short exact sequence in $QK_*(-)$, its long exact sequence is E_2 gives a long exact sequence in v_1-periodic homotopy of the K-completions, and hence the five lemma will imply that if two of the spaces satisfy the CTP, then so does the third. Because of differentials and extensions, this argument does not work for $p=2$.

Associated to a fibration $X\to Y\to Z$ is a K-homology cobar spectral sequence with

$$E_s^2 \approx \mathrm{Cotor}_s^{K_*(Z)}(K_*,K_*(Y)). \tag{5.8}$$

This spectral sequence, which generalizes that of Eilenberg–Moore [24] and is studied in [18], often, but not always, converges to $K_*(X)$. Proposition 5.10, which is implied by Ravenel [25, A1.2.8b], states that under favorable conditions it converges to $K_*(X)$ in a very strong way.

Definition 5.9. Suppose $X \to Y \to Z$ is a fibration of connected spaces whose K-homologies are free over K_*. We say the K-homology cobar spectral sequence (5.8) strongly collapses if it collapses from E^2 to the isomorphism

$$E^2_s \approx \begin{cases} 0 & s > 0, \\ K_*X & s = 0. \end{cases}$$

Proposition 5.10. *If a fibration induces a relatively injective extension sequence* (5.14) *in K-homology, then its K-homology cobar spectral sequence strongly collapses.*

Fibrations for which (5.8) strongly collapses are of some importance because of the following recent theorem of Bousfield [18].

Theorem 5.11. *Suppose the fibration $X \to Y \to Z$ induces a strongly collapsing K-homology cobar spectral sequence. Then the induced sequence $\hat{X} \to \hat{Y} \to \hat{Z}$ is of the homotopy type of a fibration.*

The notion of injective extension sequence was defined in [5, 3.1], following [12, 2.1]. However, there was a difference in the contexts. Bousfield was working over a field in [12], while [5] was working over BP_*. This difference is pointed out on [5, p. 378], where it is remarked that this "causes no change in the theory". Actually it does cause a change. One requirement for $C' \to C \to C''$ to be an injective extension sequence is that C must be injective as a C''-comodule. In the context of BP_*-modules, this will not be the case in many important examples, such as one on [5, p. 381]. Instead, Bendersky et al. [5] meant to be dealing with relatively injective modules and relatively injective extension sequences. With this change of meaning, all the statements in [5] are valid.

As in [25, A1.2.7], we make the following definition.

Definition 5.12. If Γ is a coalgebra over a commutative ring A, then an *extended Γ-comodule* is one of the form $\Gamma \otimes_A N$, where N is an A-module. A *relatively injective Γ-comodule* is a direct summand of an extended one.

The relationship that this has to the ordinary notion of injective is given in the following proposition.

Proposition 5.13 (Ravenel [25, A1.2.8]). *If a morphism $M \to N$ of Γ-comodules is monic and split over A, then any map from M to a relatively injective Γ-comodule S extends to N.*

As pointed out in [25, A1.2.9, A1.2.10, A1.2.11], the usual methods of determining Ext using injective resolutions work for relative injectives as long as we restrict attention to free modules over the ground ring.

Now we make the following definition, totally analogous to [5, 3.1].

Definition 5.14. A *relatively injective extension sequence* (RIES) is a sequence of coalgebra maps $C' \xrightarrow{f} C \xrightarrow{g} C''$ such that

- C, C', and C'' are augmented over A and projective over A;
- g is an epimorphism of A-modules;
- f is the inclusion $A \square_{C''} C \to C$;
- C is a relatively injective C''-comodule.

The following result implies that the EHP fibration

$$S^{2n-1} \to \Omega S^{2n} \to \Omega S^{4n-1} \tag{5.15}$$

induces an RIES in $K_*(-)$ and hence its K-homology cobar spectral sequence strongly collapses.

Lemma 5.16. $K_*(\Omega S^{2n}) \approx K_*(S^{2n-1}) \otimes K_*(\Omega S^{4n-1})$ *as coalgebras.*

Proof. The result is true with H_* replacing K_*. The Chern character shows that the result is true rationally. The E_2-term of the AHSS is isomorphic to

$$K_* \otimes H_*(S^{2n-1}) \otimes H_*(\Omega S^{4n-1}) \approx \Lambda_{K_*}(x) \otimes_{K_*} P_{K_*}[y],$$

where $\Lambda_{K_*}(x)$ is the exterior algebra over K_* on a class x in degree $2n-1$ and $P_{K_*}[y]$ is a polynomial algebra over K_* on a class in degree $4n-2$. A nonzero differential in the AHSS would violate the rational calculation. This proves the isomorphism in the lemma as K_*-modules. We need to show $y \in K_{4n-2}(\Omega S^{4n-1})$ is primitive. If not, the reduced coproduct has the form $\Delta(y) = x \otimes x$. But, this would imply that $\bar{x} \in K^1(\Omega S^{2n})$, the dual of x, has nontrivial square, which is a contradiction. □

Now the CTP for ΩS^{2n} is immediate from Theorem 5.11, Lemma 5.7, and the fact (noted near the end of Section 1) that ΩS^{2m-1} satisfies the CTP.

Exactly these same ingredients imply the CTP for S^{2n}, using the 2-primary fibration

$$S^{2n} \to \Omega S^{2n+1} \to \Omega S^{4n+1}.$$

This fibration induces an RIES in $K_*(-)$ by the argument used to deduce the similar statement for $BP_*(-)$ in [5, p. 388].

The proof of Theorem 1.4 for G_2 is more computational. Along with it, we prove Theorem 4.8.

Proof of Theorem 1.4 for G_2 and Theorem 4.8. We consider the diagram induced by the map $G_2 \to S^6$ (with fiber $SU(3)$)

$$\begin{array}{ccc} v_1^{-1}\pi_*(G_2) & \longrightarrow & v_1^{-1}\pi_*(S^6) \\ \downarrow & & \downarrow \\ v_1^{-1}\pi_*(\widehat{G_2}) & \longrightarrow & v_1^{-1}\pi_*(\widehat{S^6}) \end{array} \tag{5.17}$$

The analysis of $v_1^{-1}\pi_*(G_2)$ in [23], especially the figure on page 667, can be viewed as a determination of the exact sequence

$$\to v_1^{-1}\pi_* SU(3) \to v_1^{-1}\pi_* G_2 \to v_1^{-1}\pi_* S^6 \xrightarrow{\partial} v_1^{-1}\pi_{*-1} SU(3) \to . \tag{5.18}$$

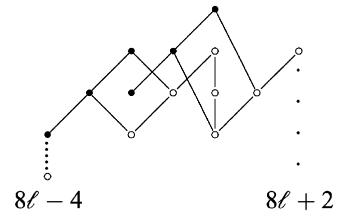

Fig. 3.

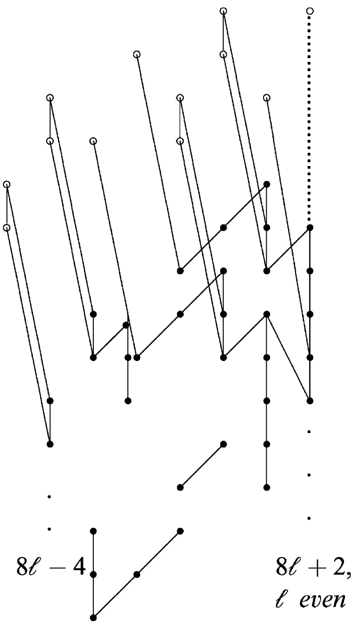

Fig. 4.

That analysis is used implicitly in the following paragraphs.

A chart for $v_1^{-1}\pi_*(SU(3))$, obtained from $v_1^{-1}\pi_*(S^3)$ (•) and $v_1^{-1}\pi_*(S^5)$ (○), is the sum in Fig. 3 with an isomorphic chart, displaced $(-1,-2)$ units. As usual, lines with negative slope represent boundary morphisms in the exact sequence of the fibration $S^3 \to SU(3) \to S^5$. The closely dotted vertical line (in $8\ell-4$) represents a nontrivial extension (multiplication by 2).

Thus $v_1^{-1}\pi_*(SU(3))$ appears as the upper part (○) in Fig. 4, in which the lower part (•) is $v_1^{-1}\pi_*(S^6)$, from [22] or [23, p. 667]. The long lines of negative slope represent ∂ in (5.18).

If ℓ is odd, the only change involves the differentials from $8\ell+2$. In this case, remove the two indicated differentials from $8\ell+2$. For exactly one of the two mod 4 congruences of odd ℓ, a differential from the bottom element in $8\ell+2$ hits the top element in $8\ell+1$. In [23, p. 668, top], it was asserted that this differential is nonzero iff $\ell \equiv 3 \bmod 4$. Although, as we shall see, this assertion is correct, there was a flaw in the argument. Diagram 4.12 of [23] does not commute, and [23, 4.6] is false. This is the same mistake that appeared for F_4/G_2 in [20] and was discussed near the end of Section 4 of this paper.

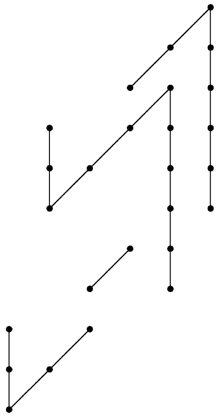

Fig. 5.

However, the Toda bracket argument of [23, p. 668] correctly implies the claim that the differential is nonzero for half the odd values of ℓ. Thus $v_1^{-1}\pi_*(G_2)$ is as in Fig. 5, with the differential from $8\ell+2$ being d_2 if ℓ is even, and d_3 for one of the mod 4 congruences of odd ℓ. Actually, the transition from Fig. 4 to Fig. 5 does not make some of the η-extensions clear, but they were established in [23].

From Fig. 4, we see that the kernel of $v_1^{-1}\pi_*(G_2) \to v_1^{-1}\pi_*(S^6)$ consists of 0, the element of order 2 in $8\ell+2$, and, for one of the mod 4 congruences of odd ℓ, the element of order 2 in $8\ell+1$. Since S^6 satisfies the CTP, (5.17) implies that the kernel of $v_1^{-1}\pi_*(G_2) \to v_1^{-1}\pi_*(\hat{G}_2)$ consists of, at most, the elements described in the preceding sentence.

Recall that the BTSS of Fig. 1 converges to $v_1^{-1}\pi_*(\hat{G}_2)$, and it must admit families of d_3-differentials annihilating all η-towers (except for a few elements at the bottom of some of the η-towers). The preceding paragraph has shown that the four parts of Fig. 5 involving $\eta \neq 0$ map nontrivially to $v_1^{-1}\pi_*(\hat{G}_2)$. Thus the pattern of differentials in the BTSS of G_2 must be as in Fig. 2 to allow for these elements of $v_1^{-1}\pi_*(\hat{G}_2)$.

The nonzero extension in dimension $8\ell+2$ in Fig. 2 for G_2 must occur because $v_1^{-1}\pi_{8\ell+1}(G_2) \to v_1^{-1}\pi_{8\ell+1}(\hat{G}_2)$ must send the element x of highest filtration in Fig. 5 nontrivially (it corresponds to the top element in $v_1^{-1}\pi_{8\ell+1}(S^6)$ in Fig. 4), and since ηx is divisible by 2, the same must be true of its image in $v_1^{-1}\pi_{8\ell+1}(\hat{G}_2)$. A similar argument implies the nontrivial extension in $t-s=8\ell+1$ in Fig. 2.

Thus $v_1^{-1}\pi_{8\ell+2}(G_2) \to v_1^{-1}\pi_{8\ell+2}(\hat{G}_2)$ is injective (since the element of order 2 maps across),

$$v_1^{-1}\pi_{8\ell+2}(\hat{G}_2) \approx \mathbb{Z}/2^{\min(6,\nu(\ell-1)+4)},$$

while $v_1^{-1}\pi_{8\ell+2}(G_2) \approx \mathbb{Z}/2^4$ if ℓ is even, and is $\mathbb{Z}/2^5$ for one odd mod 4 congruence of ℓ, and $\mathbb{Z}/2^6$ for the other. This implies that the $\mathbb{Z}/2^5$ must occur for $\ell \equiv 3 \bmod 4$, and $v_1^{-1}\pi_{8\ell+2}(G_2) \to v_1^{-1}\pi_{8\ell+2}(\hat{G}_2)$ is bijective. Thus [23, 4.1, 4.5] are valid, as are Theorems 4.8 and 1.4 for G_2. □

Finally we deduce the CTP for F_4 from the fact that (4.1) induces an RIES in $K_*(-)$, Proposition 5.10, Theorem 5.11, Lemma 5.7, and the fact that G_2 and F_4/G_2 both satisfy the CTP, the latter by Proposition 4.11. That

$$K_*(G_2) \to K_*(F_4) \to K_*(F_4/G_2)$$

is an RIES follows from Proposition 5.5 and the fact that there is a short exact sequence in $PK_1(-)$.

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